Maximally Flat Quarter-Wavelength-Coupled Transmission-Line Filters Using Q Distribution

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***Abstract*—This paper presents a technique for accurately designing maximally flat quarter-wavelength-coupled transmission-line filters using arbitrary resonant elements. The design procedure currently used, which is based on the lumped-element low-pass prototype, yields a response that is only approximately maximally flat. In addition, the current procedure results in an inaccurate prediction of the total Q for the filter. The technique presented in this paper, herein called the Q-distribution (QD) method, corrects these problems. With the QD method, the designer chooses the number and type of resonant elements and the total Q desired. In turn, the designer is provided with the individual resonator QD, which gives the designer flexibility in selecting the resonator that is most appropriate.**

***Index Terms*—Bandpass filters, Butterworth filters, maximally flat magnitude filters, Q, Q distribution, quarter-wavelength-coupled network, transmission-line resonators.**

I. Introduction

QUARTER-WAVELENGTH-COUPLED filters are the most widely used transmission-line filter. Furthermore, maximally flat filters are one of the most important types of filter designs. Accordingly, it is important to have a technique for accurately designing maximally flat quarter-wavelength-coupled filters. Currently, the approach used is only approximately correct. This paper presents a design technique, referred to as the Q-distribution (QD) method, that yields accurate results for both narrow- and wide-bandwidth designs.

A general representation of a quarter-wavelength-coupled filter is shown in Fig. 1. In this figure, each resonator, i, is represented as Q_i. This alludes to the fact that a resonator can be represented by its frequency selectivity or Q, which is the basis for the QD method. Note also in Fig. 1 that the source and load impedances are assumed to be real and equal.

The current method for designing maximally flat quarter-wavelength-coupled filters is based on the lumped-element low-pass prototype circuit [1]–[3]. This lumped-element prototype (LEP) method has two major problems. First, the response is not truly maximally flat. There are ripples in the passband which become larger with lower Q filters and with filters that employ a large number of sections. Second, the theoretical total Q does not accurately predict the actual total Q, Q_T, which is an important design parameter. Fig. 2 illustrates both of these problems. Using the LEP method, a filter was designed to have a maximally flat response with a total Q of 2.0. However, Fig. 2 clearly reveals ripples in the passband of the filter response. (S_{11} is used rather than S_{21} because it is a more sensitive indicator of ripples in the passband.) In addition, the Q of this actual response is 2.7855, which is 139.3% above the design value $Q_T = 2.0$. Also shown in Fig. 2 is the response of the same filter design using the QD method. Notice that the response using the QD method is truly maximally flat with a total Q of 2.0.

In addition to being accurate, the QD method is easy to implement. By specifying the total Q needed for the filter and the number of resonator elements that can be afforded, the QD method provides the designer with the individual resonator

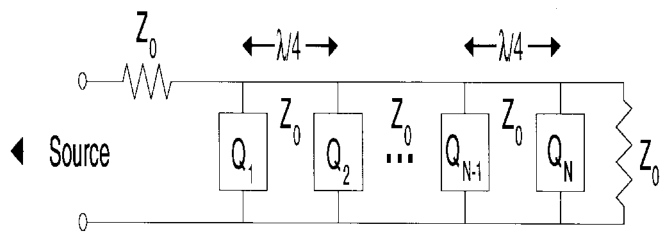

Fig. 1. Generalized quarter-wavelength coupled filter.

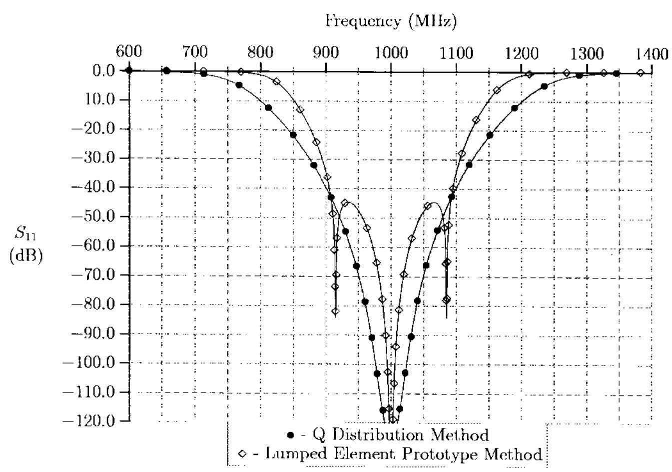

Fig. 2. Problems with the LEP method.

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Q values. With the individual resonator Q values, a designer has the freedom to choose the resonator type that is most appropriate.

The foundation for this design technique was laid by Mumford [4]. Mumford identified the mathematical form of a maximally flat response for a quarter-wavelength-coupled filter and provided tables containing admittance values for quarter-wavelength shorted-stub resonators. This paper extends the work of Mumford in several ways. First, a general design approach is provided for designing maximally flat quarter-wavelength-coupled filters. The QD method is general because it can be applied using virtually any resonator. Second, a design process is provided that is directly based on the specifications a designer has readily available, the number of resonator sections that can be implemented, and the desired total Q. Third, an accurate equation is derived that relates the individual resonator Q values to the total Q of the filter. Fourth, the method for finding a maximally flat response for quarter-wavelength-coupled filters is documented. Finally, while the LEP method has been the accepted method for designing maximally flat quarter-wavelength coupled filters, an improved design procedure is offered.

This paper begins by discussing the mathematical form of the maximally flat response for a quarter-wavelength-coupled filter. Then, the method used to create maximally flat quarter-wavelength-coupled filters, i.e., the QD Method, is described. Specifically, it is shown that using $ABCD$ matrices allows one to solve for the individual resonator Q values. These individual resonator Q values are related to the total Q of the filter. Next, calculated tables of individual resonator Q values are provided for a range of total Q values and resonator sections. Finally, to conclude, an example comparing the LEP method to the QD method is given. In this example a five-section quarter-wavelength-coupled filter that employs quarter-wavelength shorted-stub resonators is created.

II. Form of the Maximally Flat Response

There are an infinite number of maximally flat responses, but each has the requirement that all derivatives at the resonant frequency are zero. The general response for a maximally flat filter has the form [2]

$$\frac{P_0}{P_L} = 1 + (\Omega)^{2n} \tag{1}$$

where n represents the number of resonant sections. P_0 is the power available and P_L is the power delivered to the load. The function Ω depends on the total Q of the circuit and the frequency variable. For a lumped-element bandpass filter

$$\Omega = \frac{\omega_0}{\omega_2 - \omega_1}\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right) = Q_T\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right) \tag{2}$$

where ω_0 is the resonant frequency, and ω_2 and ω_1 are the upper and lower cutoff frequencies. The total loaded Q for the filter is Q_T. Thus

$$\frac{P_0}{P_L} = 1 + (Q_T)^{2n}\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)^{2n}. \tag{3}$$

It can be shown that all derivatives of (3) with respect to ω evaluated at $\omega = \omega_0$ are equal to zero.

For a quarter-wavelength-coupled filter, Mumford [4] identified the maximally flat form as

$$\frac{P_0}{P_L} = 1 + K_n \frac{\cos^{2n}\theta}{\sin^2\theta} \tag{4}$$

where $\theta = \pi\omega/2\omega_0 = \pi f/2f_0$, n is the number of resonators, and K_n is a constant which depends on the individual resonator values. For the special case of using quarter-wavelength shorted-stubs as resonators, Mumford found that

$$K_n = \left[\frac{y_{q1}(y_{q2}+2)\cdots(y_{qn}+2)}{2}\right]^2 \tag{5}$$

where y_{qr} is the normalized characteristic admittance for the rth stub such that

$$y_{qr} = \frac{Y_{qr}}{Y_0} \tag{6}$$

where Y_0 is the admittance of the quarter-wavelength coupling line. It can be shown that the derivatives of (4) with respect to ω evaluated at $\omega = \omega_0$ are zero.

III. Solving for the Maximally Flat Response for a Quarter-Wavelength-Coupled Filter

Solving for the maximally flat response is a matter of comparing the general response for an n-section quarter-wavelength coupled filter to the maximally flat form given in (4). Unfortunately, there is no direct way to find the general response. As a result, the problem is approached by assuming that all of the individual resonator elements are quarter-wavelength shorted stubs and comparing the filter response to the maximally flat form.

To find the response for an n-section quarter-wavelength-coupled filter with quarter-wavelength shorted-stub resonators, $ABCD$ matrices are used. The response is identified from the composite $ABCD$ matrix. Since each resonant section and each quarter-wavelength transmission-line section have an $ABCD$ matrix, to find a usable composite $ABCD$ matrix it is necessary to multiply the $2n$ $ABCD$ matrices together and then collect terms where each term is in a trigonometric form. Although this approach is valid for finding the response, as the number of resonator elements used (n) increases, the size and complexity of the composite $ABCD$ matrix becomes unmanageably large. One way to simplify the analysis is to introduce the variable $p = -j\cot\theta$ [5]. This effectively converts the terms in the composite $ABCD$ matrix from a trigonometric form to a polynomial form which significantly reduces the mathematical complexity. Another simplification is achieved by considering the symmetry present in the maximally flat solution. Specifically, the maximally flat solution imposes a symmetrical filter structure. By using this structural symmetry, the number of variables and equations are effectively reduced by a factor of two.

For quarter-wavelength shorted-stub resonators, comparing the response of an n-section filter to the maximally flat form results in a solution for the normalized admittance values for the n shorted-stub resonators. To generalize these results, the admittance values are converted to individual resonator Q values, so that a designer can apply these results to any arbitrary resonator. Finally, a relation is needed between the determined individual resonator Q values and the total Q of the filter. This relationship is found by first using (5) to relate the normalized admittance values to K_n, the coefficient of the maximally flat form, and then relating K_n to the total Q. Unfortunately, a closed-form relation does not exist between K_n and Q_T. However, a numerical solution can be found, and from the graphical form of this numerical solution, an extremely accurate "linear" approximation is derived that relates K_n to Q_T in equation form.

The steps described above form the outline to this section. These same steps are used to generate the tables in Section IV that are used by designers to create maximally flat quarter-wavelength-coupled filters. Also in this section, an example is provided for implementing this method to create a maximally flat five-section filter. This five-section filter example will be used again in Section VI when the responses generated by the QD method and the LEP method are compared.

A. *ABCD Matrices and the Maximally Flat Form*

From the composite $ABCD$ matrix of a filter, the transducer loss $L_t = P_0/P_L$ for a lossless transmission-line network that has both source and load resistance equal to Z_0 is given by

$$\frac{P_0}{P_L} = 1 + \frac{1}{4}\left[(A-D)^2 - \left(\frac{B}{Z_0} - Z_0 C\right)^2\right]. \tag{7}$$

Furthermore, it can be shown that if the network is symmetric, then $A = D$ and

$$\frac{P_0}{P_L} = 1 + \frac{1}{4}\left[-\left(\frac{B}{Z_0} - Z_0 C\right)^2\right]. \tag{8}$$

Equating (8) to the form for a maximally flat quarter-wavelength-coupled filter in (4) yields the following relation:

$$\frac{P_0}{P_L} = 1 + \frac{1}{4}\left[-\left(\frac{B}{Z_0} - Z_0 C\right)^2\right] = 1 + K_n \frac{\cos^{2n}\theta}{\sin^2\theta}. \tag{9}$$

B. *Analysis Using Quarter-Wavelength Shorted-Stub Resonators*

Since it is not possible to represent a general resonator as an $ABCD$ matrix, it is necessary to choose a resonator that can be represented as an $ABCD$ matrix. Because of its simple $ABCD$ matrix form, the quarter-wavelength shorted-stub resonator is chosen. The $ABCD$ matrix for a parallel quarter-wavelength shorted-stub resonator of characteristic admittance Y_{qr} is

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -jY_{qr}\cot\theta & 1 \end{bmatrix} \tag{10}$$

where $\theta = (\pi/2)(\omega/\omega_0)$. The $ABCD$ matrix for a quarter-wavelength section of transmission line of characteristic admittance Y_0 is

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \cos\theta & j\dfrac{\sin\theta}{Y_0} \\ jY_0\sin\theta & \cos\theta \end{bmatrix}. \tag{11}$$

1) General Form of Solution: For an n-section quarter-wavelength-coupled filter that uses quarter-wavelength shorted-stub resonators, the composite $ABCD$ matrix is given by

$$\begin{aligned}\begin{bmatrix} A & B \\ C & D \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ -jY_{q1}\cot\theta & 1 \end{bmatrix}\begin{bmatrix} \cos\theta & j\dfrac{\sin\theta}{Y_0} \\ jY_0\sin\theta & \cos\theta \end{bmatrix} \\ &\cdot \begin{bmatrix} 1 & 0 \\ -jY_{q2}\cot\theta & 1 \end{bmatrix}\cdots\begin{bmatrix} \cos\theta & j\dfrac{\sin\theta}{Y_0} \\ jY_0\sin\theta & \cos\theta \end{bmatrix} \\ &\cdot \begin{bmatrix} 1 & 0 \\ -jY_{qn}\cot\theta & 1 \end{bmatrix}.\end{aligned} \tag{12}$$

Multiplying the matrices in (12) and using (8), the general form of the response for this n-section filter is found to be

$$\begin{aligned}\frac{P_0}{P_L} =& 1 + a_0\frac{\cos^{2n}\theta}{\sin^2\theta} + a_1\cos^{2n}\theta \\ &+ a_2\cos^{2(n-1)}\theta\sin^2\theta + ,\cdots, \\ &+ a_{n-1}\cos^2\theta\sin^{2(n-1)}\theta\end{aligned} \tag{13}$$

where

$$a_i = f(Y_{q1},\cdots,Y_{qn}) \tag{14}$$

and Y_{qi} is the admittance for the ith quarter-wavelength shorted-stub resonator.

Finally, the parameters, i.e., admittance values, that make the filter maximally flat are found by comparing the response (13) to the maximally flat form (9). Equating the two forms results in the following system of equations:

$$\begin{bmatrix} a_1 = 0 \\ a_2 = 0 \\ \vdots \\ a_{n-1} = 0 \end{bmatrix} \tag{15}$$

with $a_0 = K_n$. Since there are $n-1$ equations and n unknowns $(Y_{q1},\cdots,Y_{qn})$, the system is under-determined. Thus, it is necessary to choose the value for one Y_{qi} and then use (15) to solve for the other $n-1$ values. It should be noted that the choice of a value for Y_{qi} is not completely arbitrary because the value chosen fixes the total Q for the filter. To see this, note that if the equation for a_0 is added to the system of equations, there would be n equations and n unknowns, and thus the system is fully determined. The coefficient a_0 is the same as the coefficient of the maximally flat form K_n, and K_n is directly related to the total Q for the filter. The relationship between K_n and total Q will be discussed in Section III-E. Thus, the choice of a value for Y_{qi} effectively fixes a_0 which sets the total Q of the filter.

2) Analysis of a Three-Section Filter: As a simple example of the above procedure, the characteristic admittance values are determined for a maximally flat three-section filter using quarter-wavelength shorted-stub resonators. First, the composite $ABCD$ matrix is created. Three quarter-wavelength shorted stubs coupled by quarter-wavelength line sections yield the composite $ABCD$ matrix:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -jY_{q1}\cot\theta & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & j\dfrac{\sin\theta}{Y_0} \\ jY_0\sin\theta & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ -jY_{q2}\cot\theta & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & j\dfrac{\sin\theta}{Y_0} \\ jY_0\sin\theta & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ -jY_{q3}\cot\theta & 1 \end{bmatrix}. \quad (16)$$

Multiplying the matrices yields (17), as shown at the bottom of the page.

To find the transducer loss, (17) is substituted into (7) to obtain

$$\frac{P_0}{P_L} = 1 + a_0\frac{\cos^4\theta}{\sin^2\theta} + a_1\cos^4\theta + a_2\cos^2\theta\sin^2\theta \quad (18)$$

where

$$\begin{aligned} a_0 &= \frac{1}{4}\left(\frac{Y_{q1}+Y_{q2}+Y_{q3}}{Y_0} + \frac{Y_{q1}Y_{q2}+2Y_{q1}Y_{q3}+Y_{q2}Y_{q3}}{Y_0^2} + \frac{Y_{q1}Y_{q2}Y_{q3}}{Y_0^3}\right)^2 \\ a_1 &= \frac{1}{4}\left(\frac{2Y_{q1}-2Y_{q3}}{Y_0} + \frac{Y_{q2}Y_{q3}-Y_{q1}Y_{q2}}{Y_0^2}\right)^2 \\ a_2 &= \frac{1}{4}\left(\frac{Y_{q2}-Y_{q1}-Y_{q3}}{Y_0}\right)^2. \end{aligned} \quad (19)$$

The solution for Y_{q1}, Y_{q2}, and Y_{q3} that gives a maximally flat response is found by simultaneously solving

$$\begin{aligned} a_1 &= 0 \\ a_2 &= 0 \end{aligned} \quad (20)$$

and is found to be

$$\begin{aligned} Y_{q2} &= 2Y_{q1} \\ Y_{q3} &= Y_{q1}. \end{aligned} \quad (21)$$

Substituting (21) into a_0 yields

$$a_0 = 4\frac{Y_{q1}^2}{Y_0^2} + 12\frac{Y_{q1}^3}{Y_0^3} + 13\frac{Y_{q1}^4}{Y_0^4} + 6\frac{Y_{q1}^5}{Y_0^5} + 5\frac{Y_{q1}^6}{Y_0^6}. \quad (22)$$

Likewise substituting (21) into

$$K_3 = \left[\frac{\dfrac{Y_{q1}}{Y_0}\left(\dfrac{Y_{q2}}{Y_0}+2\right)\left(\dfrac{Y_{q3}}{Y_0}+2\right)}{2}\right]^2 \quad (23)$$

yields

$$K_3 = 4\frac{Y_{q1}^2}{Y_0^2} + 12\frac{Y_{q1}^3}{Y_0^3} + 13\frac{Y_{q1}^4}{Y_0^4} + 6\frac{Y_{q1}^5}{Y_0^5} + 5\frac{Y_{q1}^6}{Y_0^6}. \quad (24)$$

Thus, $a_0 = K_3$ as predicted by (4).

C. Simplifications to General Analysis

Although the method described in the previous section is valid for any number of sections, as the number of sections becomes large, the complexity of the resulting matrix and, consequently, the system of equations (15) becomes unmanageable. However, two simplifications significantly reduce the mathematical complexity of the problem. First, the mathematical complexity of (12) is lessened by converting this trigonometric form to a polynomial form. Second, since each section adds one equation to the system of equations in (15), an n-section filter requires the simultaneous solution of $n-1$ equations. By making use of the filter-structure symmetry required by the maximally flat solution, the number of equations is effectively halved.

1) p Substitution: Mathematically, it is easier to deal with a polynomial function than a trigonometric function. Converting the form of the problem from a trigonometric to polynomial form is accomplished by using the substitution [5], [6]

$$p = -j\cot\theta. \quad (25)$$

Thus, (10) becomes

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ pY_{qr} & 1 \end{bmatrix} \quad (26)$$

and (11) becomes

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = j\sin\theta \begin{bmatrix} p & \dfrac{1}{Y_0} \\ Y_0 & p \end{bmatrix}. \quad (27)$$

An n-section quarter-wavelength shorted-stub resonator filter can be represented as

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = (j\sin\theta)^{n-1} \begin{bmatrix} 1 & 0 \\ pY_{q1} & 1 \end{bmatrix} \begin{bmatrix} p & \dfrac{1}{Y_0} \\ Y_0 & p \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ pY_{q2} & 1 \end{bmatrix} \cdots \begin{bmatrix} p & \dfrac{1}{Y_0} \\ Y_0 & p \end{bmatrix} \begin{bmatrix} 1 & 0 \\ pY_{qn} & 1 \end{bmatrix} \quad (28)$$

or, after multiplying the matrices for n odd, the composite $ABCD$ simplifies to (29) as shown at the bottom of the following page. For n even, the composite $ABCD$ simplifies

$$\begin{bmatrix} \left(1+\dfrac{Y_{q2}+2Y_{q3}}{Y_0}+\dfrac{Y_{q2}Y_{q3}}{Y_0^2}\right)\cos^2\theta - \sin^2\theta & j\dfrac{2Y_0+Y_{q2}}{Y_0^2}\cos\theta\sin\theta \\ j(2Y_0+Y_{q1}+Y_{q3})\cos\theta\sin\theta - j\left(Y_{q1}+Y_{q2}+Y_{q3}+\dfrac{Y_{q1}Y_{q2}+2Y_{q1}Y_{q3}+Y_{q2}Y_{q3}}{Y_0}+\dfrac{Y_{q1}Y_{q2}Y_{q3}}{Y_0^2}\right)\cot\theta\cos^2\theta & \left(1+\dfrac{2Y_{q1}+Y_{q2}}{Y_0}+\dfrac{Y_{q1}Y_{q2}}{Y_0^2}\right)\cos^2\theta - \sin^2\theta \end{bmatrix} \quad (17)$$

as shown in (30), at the bottom of the page, where $s = (n+1)/2$ for n odd and $s = n/2$ for n even, and

$$\begin{aligned} A_i &= f\left(\frac{Y_{q1}}{Y_0}, \cdots, \frac{Y_{qn}}{Y_0}\right) \\ B_i &= f\left(\frac{Y_{q1}}{Y_0}, \cdots, \frac{Y_{qn}}{Y_0}\right) \\ C_i &= f\left(\frac{Y_{q1}}{Y_0}, \cdots, \frac{Y_{qn}}{Y_0}\right) \\ D_i &= f\left(\frac{Y_{q1}}{Y_0}, \cdots, \frac{Y_{qn}}{Y_0}\right). \end{aligned} \tag{31}$$

The maximally flat form is obtained by setting

$$A_i = D_i$$

and

$$Y_0 B_i = \frac{C_i}{Y_0}, \qquad \forall\, i = 1, \cdots, s-1. \tag{32}$$

Once again, the system is under-determined with $n-1$ equations and n unknowns. There is also one unmatched coefficient, C_s. By arbitrarily choosing one of the admittance values Y_{qi}/Y_0, the other admittance values, Y_{qj}/Y_0 for $j \neq i$, can be found using (32). The value chosen for Y_{qi}/Y_0 determines the coefficient C_s, which is directly related to the total Q of the filter.

To show that (32) is the maximally flat solution, note that applying (32) to (7) sets all polynomial terms less than p^n to zero, which yields the following result for the transducer loss:

$$\begin{aligned} \frac{P_0}{P_L} &= 1 + \frac{1}{4}\left\{-\left[(j \sin\theta)^{n-1} \frac{(-1)C_s p^n}{Y_0}\right]^2\right\} \\ &= 1 + \frac{-j^{4n-2}C_s^2}{4Y_0^2} \cot^{2n}\theta \sin^{2(n-1)}\theta \end{aligned} \tag{33}$$

or

$$\frac{P_0}{P_L} = 1 + \frac{C_s^2}{4Y_0^2}\frac{\cos^{2n}\theta}{\sin^2\theta} \tag{34}$$

which is of the maximally flat form given by (4) with

$$K_n = \frac{C_s^2}{4Y_0^2}. \tag{35}$$

In Section III-E it will be shown how K_n and thus, C_s, relate to the total Q of the filter.

2) Using Symmetry to Simplify: A symmetrical network structure implies that the characteristic admittance for the first stub of a quarter-wavelength shorted-stub filter is equal to the admittance of the last stub, the second stub admittance is equal to the second to last stub admittance, and so on. Thus

$$\begin{bmatrix} Y_{q1} = Y_{qn} & \\ Y_{q2} = Y_{q(n-1)} & \\ \vdots & \\ Y_{q[(n-1)/2]} = Y_{q[(n+1)/2]}, & \text{for } n \text{ odd} \\ Y_{q(n/2)} = Y_{q[(n+2)/2]}, & \text{for } n \text{ even} \end{bmatrix}. \tag{36}$$

Symmetry also imposes the condition on the composite $ABCD$ matrix that the A element must equal the D element. That is

$$A_i = D_i \quad \forall\, i = 1, \cdots, s-1. \tag{37}$$

As a result, by imposing symmetry, both the number of variables (36) and the number of equations (37) are halved. Thus, the maximally flat solution is found by solving

$$Y_0 B_i = C_i / Y_0 \qquad \forall\, i = 1, \cdots, s-1. \tag{38}$$

A maximally flat response has a symmetrical solution. That is, the solution of (32) yields the condition given in (36). Although a formal proof cannot be offered, the validity of this assertion can be demonstrated. The $ABCD$ matrix of the form given by (30) for a four-section quarter-wavelength-coupled filter that uses quarter-wavelength shorted-stub resonators is shown in (39), at the bottom of the following page. Setting $A = D$ gives the following two equations:

$$3 + Y_{q2} + Y_{q4} = 3 + Y_{q1} + Y_{q3}$$

and

$$\begin{aligned} &1 + Y_{q2} + 2Y_{q3} + 3Y_{q4} + Y_{q2}Y_{q3} + 2Y_{q2}Y_{q4} \\ &\quad + 2Y_{q3}Y_{q4} + Y_{q2}Y_{q3}Y_{q4} \\ &\qquad = 1 + 3Y_{q1} + 2Y_{q2} + Y_{q3} + 2Y_{q1}Y_{q2} + 2Y_{q1}Y_{q3} \\ &\qquad\quad + Y_{q2}Y_{q3} + Y_{q1}Y_{q2}Y_{q3} \end{aligned} \tag{40}$$

which simply leads to

$$Y_{q1} - Y_{q4} = Y_{q2} - Y_{q3}$$

and

$$\begin{aligned} 0 = &3(Y_{q1} - Y_{q4}) + (Y_{q2} - Y_{q3}) + Y_{q2}Y_{q3}(Y_{q1} - Y_{q4}) \\ &+ 2Y_{q3}(Y_{q1} - Y_{q4}) + 2Y_{q2}(Y_{q1} - Y_{q4}) \end{aligned} \tag{41}$$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = (j \sin\theta)^{n-1} \begin{bmatrix} 1 + A_1p^2 + A_2p^4 + \cdots + A_{s-1}p^{n-1} & B_1p + B_2p^3 + \cdots + B_{s-1}p^{n-2} \\ C_1p + C_2p^3 + \cdots + C_{s-1}p^{n-2} + C_sp^n & 1 + D_1p^2 + D_2p^4 + \cdots + D_{s-1}p^{n-1} \end{bmatrix} \tag{29}$$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = (j \sin\theta)^{n-1} \begin{bmatrix} A_1p^1 + A_2p^3 + \cdots + A_{s-1}p^{n-1} & 1 + B_1p^2 + B_2p^4 + \cdots + B_{s-1}p^{n-2} \\ 1 + C_1p^2 + C_2p^4 + \cdots + C_{s-1}p^{n-2} + C_sp^n & D_1p^1 + D_2p^3 + \cdots + D_{s-1}p^{n-1} \end{bmatrix} \tag{30}$$

which leads to the symmetry conditions

$$\begin{aligned} Y_{q1} &= Y_{q4} \\ Y_{q2} &= Y_{q3}. \end{aligned} \tag{42}$$

A similar analysis can be performed on any n-section quarter-wavelength-coupled filter with the same result. Specifically, with $A = D$, as required for a maximally flat response, the solution set is the symmetry conditions (36). Conversely, applying the symmetry conditions (36) results in $A = D$, which is required for a maximally flat response.

D. Generalizing Results from Quarter-Wavelength Shorted-Stub Resonators to Arbitrary Resonators

At this point, the maximally flat response has been created for a filter with quarter-wavelength shorted-stub resonators by using the symmetry conditions (36) and solving (38). To generalize these results to any resonator, it is necessary to convert the admittance values Y_{qi} into Q values for each individual resonator Q_i. This is accomplished by using the relationship for the Q of a single quarter-wavelength shorted stub on a transmission line with admittance Y_0 [7], [8] as follows:

$$Q_i = \frac{\pi Y_{qi}}{8Y_0} = \frac{\pi Z_0}{8Z_{qi}}. \tag{43}$$

Once the individual resonator Q values necessary for a maximally flat response are known, the designer may choose any resonator that operates at the central frequency of the filter ω_0. The designer only needs to know how the Q of the resonator relates to the parameters of the resonator. For example, for a series-connected lumped-element resonator (which has parameters L and C), the Q is

$$Q_i = \frac{\omega_0 L}{2Z_0} \tag{44}$$

where ω_0 is the resonant frequency and L is the inductance. C is found by using the resonant condition

$$C = \frac{1}{\omega_0{}^2 L}. \tag{45}$$

The relationship between the Q of the resonator and the parameters of the resonator can be found mathematically either by using the fundamental definition of Q:

$$Q = \omega_0 \frac{\text{Peak Energy Stored}}{\text{Average Power Lost}} \tag{46}$$

or as follows (for series-connected elements):

$$Q = \left[\frac{\omega}{2R_{\text{ser}}} \frac{\partial X}{\partial \omega}\right]_{\omega=\omega_0} \tag{47}$$

and as follows (for parallel-connected elements):

$$Q = \left[\frac{\omega}{2G_{\text{par}}} \frac{\partial B}{\partial \omega}\right]_{\omega=\omega_0} \tag{48}$$

where $R_{\text{ser}} = 2Z_0$ and $G_{\text{par}} = 2Y_0$. These expressions are derived from the fundamental definition of Q [9]. These relationships are also given in [7, p. 414]. If a mathematical formulation is not possible, the Q of an individual resonator can be determined from measured S parameter data [10] or from another Q measurement technique [11]–[14].

E. Relation Between Individual Q Values and Total Q

The previous discussion has provided the designer with a method for determining the valid combinations of individual resonator Q values for a given number of sections, n, that yield a maximally flat response. The final step is to relate this QD to the total Q of the filter, Q_T. First, note that in the process of finding the QD using the under-determined system of equations (38), one coefficient remained, C_s. Choosing a value for one of the resonators fixed the value of C_s, and thus by (35) fixed the value of K_n, the coefficient of the maximally flat form. Mumford [4] found a direct relation between K_n and the normalized admittance values for the specific case of a quarter-wavelength shorted-stub filter (5). This relationship may be extended to arbitrary resonators by relating the admittance values to individual Q values (43), giving the relation between K_n and the QD as

$$K_n = \left(\frac{8}{\pi}\right)^{2n} \left[\frac{Q_1\left(Q_2 + \frac{\pi}{4}\right)\cdots\left(Q_n + \frac{\pi}{4}\right)}{2}\right]^2. \tag{49}$$

Since K_n is the coefficient in the maximally flat form of the quarter-wavelength-coupled filter in (4), to relate the individual QD to Q_T, it is necessary to only relate K_n and Q_T. This is obtained from the general form of the maximally flat response for a quarter-wavelength-coupled filter. Equation (4) may be rewritten as

$$\frac{P_L}{P_0} = \frac{1}{1 + K_n \dfrac{\cos^{2n}\theta}{\sin^2\theta}} \tag{50}$$

where $\theta = \pi\omega/2\omega_0$. The total Q is given by

$$Q_T = \frac{\omega_0}{\omega_2 - \omega_1} \tag{51}$$

where ω_0 is the resonant frequency, and ω_2 and ω_1 are the upper and lower cutoff frequencies. The lower and upper

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = (j\sin\theta)^{4-1} \begin{bmatrix} \begin{array}{c}(3 + Y_{q2} + Y_{q4})p + \\ (1 + Y_{q2} + 2Y_{q3} + 3Y_{q4} + Y_{q2}Y_{q3} + \\ 2Y_{q2}Y_{q4} + 2Y_{q3}Y_{q4} + Y_{q2}Y_{q3}Y_{q4})p^3\end{array} & 1 + B_1 p^2 \\ 1 + C_1 p^2 + C_2 p^4 & \begin{array}{c}(3 + Y_{q1} + Y_{q3})p + \\ (1 + 3Y_{q1} + 2Y_{q2} + Y_{q3} + 2Y_{q1}Y_{q2} + \\ 2Y_{q1}Y_{q3} + Y_{q2}Y_{q3} + Y_{q1}Y_{q2}Y_{q3})p^3\end{array} \end{bmatrix} \tag{39}$$

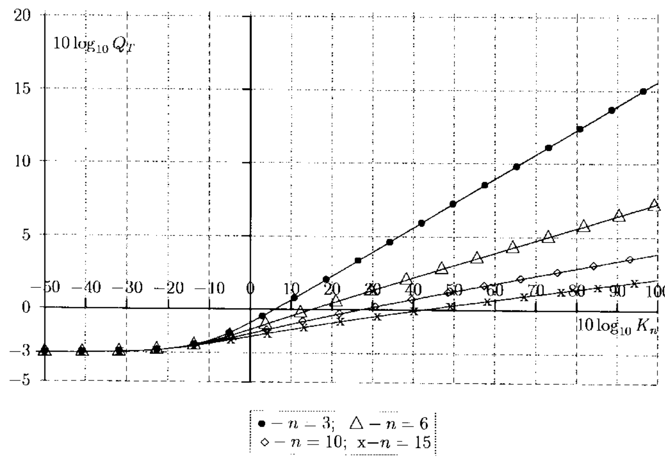

Fig. 3. Q_T versus K_n for different numbers of sections (n).

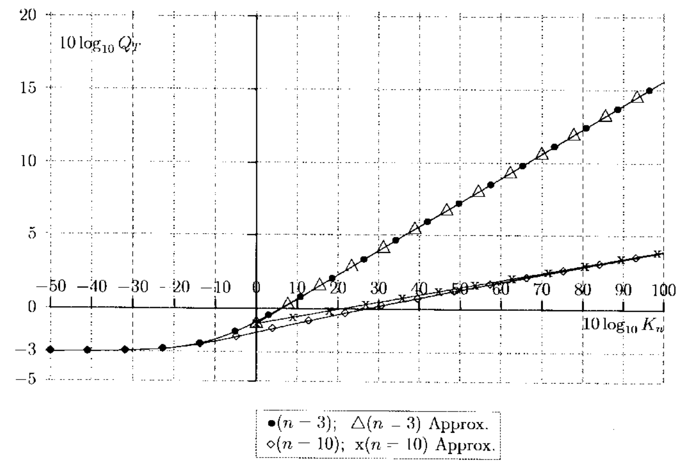

Fig. 4. Comparison of actual to linear approximation for Q_T versus K_n.

frequencies are calculated based on where $P_L/P_0 = 1/2$. Using (50), this occurs when

$$\sin^2\left(\frac{\pi\omega}{2\omega_0}\right) = K_n \cos^{2n}\left(\frac{\pi\omega}{2\omega_0}\right). \tag{52}$$

A generalized closed-form solution of (52) is not available. However, a numerical solution is possible, and an approximate closed-form solution is derived that is very accurate for most applications. Numerically solving (52) for a given n, ω_0, and K_n, yields two solutions for ω, ω_2, and ω_1, which when substituted into (51) yields the value for Q_T. The QD tables presented in Section IV are indexed by this numerical solution of Q_T.

To derive an approximate closed-form expression for Q_T in terms of K_n, it is necessary to begin with the numerical solution. From a graph of the numerical solution of K_n versus Q_T for several values of n on a $\log_{10}$–$\log_{10}$ scale, as shown in Fig. 3, a definite trend can be observed. First, notice that for all values of n, all curves start at a constant value of $10\log_{10} Q_T = -3$. At some point, these curves bend to a linear slope, eventually reaching a slope of $1/2n$.

Since the point at which the curves bend to a linear slope is below the range of usable K_n values, $K_n < 1$, a linear equation can be developed for the usable portion of the curve in the following form:

$$y = mx + b \tag{53}$$

where m is the slope, and b is the intercept. By using two points on each curve that have a large value of K_n, it is found that all curves have a common intercept point $b = 10\log_{10}(\pi/4)$ and a slope of $m = 1/2n$. This leads to the following linear approximation for the upwardly sloping portion of the curve as a function of n:

$$10\log_{10}(Q_T) = \frac{10\log_{10}(K_n)}{2n} + 10\log_{10}\left(\frac{\pi}{4}\right). \tag{54}$$

Taking the antilog of both sides yields the following closed-form approximate relation between K_n and Q_T:

$$Q_T = \frac{\pi}{4} K_n^{1/2n}. \tag{55}$$

Fig. 4 compares the linear approximation given by (54) to the exact numerical solution given in Fig. 3 for three sections ($n = 3$) and ten sections ($n = 10$). Notice that for the widely used range of Q_T values (i.e., $Q_T > 1.0$), there is excellent agreement. The largest deviation occurs for small values of K_n, and increases with the number of resonators.

Substituting (49) in (55) yields

$$Q_T = 2\left[\frac{Q_1\left(Q_2 + \frac{\pi}{4}\right)\cdots\left(Q_n + \frac{\pi}{4}\right)}{2}\right]^{1/n}. \tag{56}$$

Equation (56) is a very important result because it relates the individual QD to the total Q of the filter, Q_T. It is extremely accurate, achieving greater accuracy at higher values of Q_T and lower values of n. For example, a six-resonator-section filter that has an actual $Q_T = 50.0$ will have a $Q_T = 50.001$ using (56), and an actual $Q_T = 1.0$ will have a $Q_T = 1.048$ with (56). To further support the validity of (56), the QD tables in Section IV provide a column for both actual values of Q_T and the approximate values of Q_T given by (56).

As an additional check to the solutions given in (55) and (56), the results obtained for one resonant section are examined. Assuming this resonator is a quarter-wavelength shorted-stub, Mumford's relation for K_n (5) for $n = 1$ is

$$K_1 = \left(\frac{Y_{q1}}{2Y_0}\right)^2. \tag{57}$$

Substituting (57) into (55) with $n = 1$ yields

$$Q_T = \frac{\pi Y_{q1}}{8Y_0} \tag{58}$$

which is the expression for the Q of a single quarter-wavelength shorted-stub resonator, as given in (43). Moreover, for one resonant section, (56) simplifies to

$$Q_T = Q_1 \tag{59}$$

as expected.

TABLE I
PROCEDURE FOR FINDING THE QD AND Q_T

Step	Procedure
1	Choose number of sections, n
2	Create composite $ABCD$ matrix using (28)
3	For each matrix element, collect polynomial terms
4	Apply symmetry conditions (36)
5	Choose one admittance value, Y_{qi}
6	Solve for other admittance values using (32)
7	Convert admittance values, Y_{qi}, to Q values, Q_i, using (43)
8	Find Q_T from individual Q values using (56)

F. Relation to the General Maximally Flat Form

The result given in (55) can be used to show that the form of quarter-wavelength-coupled filters is indeed a general maximally flat form, i.e., of the form given in (1). Solving (55) for K_n in terms of Q_T yields

$$K_n = \left[\frac{4}{\pi} Q_T\right]^{2n}. \tag{60}$$

Substituting (60) into (4) yields

$$\frac{P_0}{P_L} = 1 + \frac{\left(\frac{4}{\pi} Q_T \cos\theta\right)^{2n}}{\sin^2\theta} \tag{61}$$

which is of the form given in (1) with

$$\Omega = \frac{\frac{4}{\pi} Q_T \cos\theta}{\sin^{1/n}\theta}. \tag{62}$$

G. Summary of Procedure for Finding QD

Table I summarizes the procedure for finding the QD of individual resonators and total Q for a given number of sections, n.

H. Example: Five-Section Filter ($n = 5$)

Following the summary given in Table I, an example is given for designing a maximally flat five-section filter using the QD method. From (28), the $ABCD$ matrix for a five-section quarter-wavelength shorted-stub resonator filter is represented as

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = (j\sin\theta)^4 \begin{bmatrix} 1 & 0 \\ pY_{q1} & 1 \end{bmatrix} \begin{bmatrix} p & \frac{1}{Y_0} \\ Y_0 & p \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ pY_{q2} & 1 \end{bmatrix} \cdots \begin{bmatrix} p & \frac{1}{Y_0} \\ Y_0 & p \end{bmatrix} \begin{bmatrix} 1 & 0 \\ pY_{q5} & 1 \end{bmatrix}. \tag{63}$$

Multiplying (63) and collecting polynomial terms gives the simplified composite $ABCD$ matrix:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = (j\sin\theta)^4 \cdot \begin{bmatrix} 1 + A_1p^2 + A_2p^4 & B_1p + B_2p^3 \\ C_1p + C_2p^3 + C_3p^5 & 1 + D_1p^2 + D_2p^4 \end{bmatrix}. \tag{64}$$

If the symmetry conditions (36) are applied such that $Y_{q1} = Y_{q5}$ and $Y_{q2} = Y_{q4}$, the coefficients of (64) are

$$\begin{aligned}
A_1 &= D_1 = 6 + \frac{4Y_{q1} + 4Y_{q2} + 2Y_{q3}}{Y_0} + \frac{2Y_{q1}Y_{q2} + Y_{q2}Y_{q3}}{Y_0^2} \\
A_2 &= D_2 = 1 + \frac{4Y_{q1} + 4Y_{q2} + 2Y_{q3}}{Y_0} \\
&\quad + \frac{2Y_{q2}^2 + 6Y_{q1}Y_{q2} + 4Y_{q1}Y_{q3} + 3Y_{q2}Y_{q3}}{Y_0^2} \\
&\quad + \frac{2Y_{q1}^2Y_{q2} + Y_{q2}^2Y_{q3} + 4Y_{q1}Y_{q2}Y_{q3}}{Y_0^3} \\
&\quad + \frac{Y_{q1}Y_{q2}^2Y_{q3}}{Y_0^4} \\
B_1 &= \frac{4}{Y_0} + 2\frac{Y_{q2}}{Y_0^2} \\
B_2 &= \frac{4}{Y_0} + \frac{6Y_{q2} + 4Y_{q3}}{Y_0^2} + \frac{2Y_{q2}^2 + 4Y_{q2}Y_{q3}}{Y_0^3} + \frac{Y_{q2}^2Y_{q3}}{Y_0^4} \\
C_1 &= 4Y_0 + 2Y_{q1} + Y_{q3} \\
C_2 &= 4Y_0 + 12Y_{q1} + 6Y_{q2} + 2Y_{q3} \\
&\quad + \frac{4Y_{q1}^2 + 8Y_{q1}Y_{q2} + 4Y_{q1}Y_{q3} + 2Y_{q2}Y_{q3}}{Y_0} \\
&\quad + \frac{2Y_{q1}^2Y_{q2} + 2Y_{q1}Y_{q2}Y_{q3}}{Y_0^2} \\
C_3 &= 2Y_{q1} + 2Y_{q2} + Y_{q3} \\
&\quad + \frac{4Y_{q1}^2 + 2Y_{q2}^2 + 8Y_{q1}Y_{q2} + 4Y_{q1}Y_{q3} + 2Y_{q2}Y_{q3}}{Y_0} \\
&\quad + \frac{4Y_{q1}Y_{q2}^2 + 4Y_{q1}^2Y_{q3} + Y_{q2}^2Y_{q3} + 6Y_{q1}Y_{q2}Y_{q3}}{Y_0^2} \\
&\quad + \frac{2Y_{q1}^2Y_{q2}^2 + 4Y_{q1}^2Y_{q2}Y_{q3} + 2Y_{q1}Y_{q2}^2Y_{q3}}{Y_0^3} \\
&\quad + \frac{Y_{q1}^2Y_{q2}^2Y_{q3}}{Y_0^4}.
\end{aligned} \tag{65}$$

Setting $B_1Y_0 = C_1/Y_0$ and $B_2Y_0 = C_2/Y_0$ yields the following relation between $Y_{q1}/Y_0 = y_{q1}$, $Y_{q2}/Y_0 = y_{q2}$, and $Y_{q3}/Y_0 = y_{q3}$:

$$\begin{aligned}
y_{q2} &= \frac{-1 + 3y_{q1} + \sqrt{1 + 10y_{q1} + 5y_{q1}^2}}{2} \\
y_{q3} &= -1 + y_{q1} + \sqrt{1 + 10y_{q1} + 5y_{q1}^2}.
\end{aligned} \tag{66}$$

TABLE II
VALUES FOR THE FIVE-SECTION FILTER EXAMPLE

	Value
y_{q1} (chosen)	1.2680
y_{q2}	0.4141
y_{q3}	0.3075
C_3	225.136
K_5	12671.6
Q_1	0.3097
Q_2	0.9483
Q_3	1.2772
Q_T	2.0
Q_T (approx.)	2.0201

As expected, there are two equations and three unknowns. Therefore, a value for one of the Y_{qi} may be arbitrarily chosen. The choice of Y_{qi} determines the value of C_3 and, thus, ultimately the total Q for the filter. Then, (43) is used to convert these admittance values to the following individual resonator Q values:

$$Q_1 = \frac{\pi}{8} y_{q1}$$
$$Q_2 = \frac{\pi}{8} y_{q2}$$
$$Q_3 = \frac{\pi}{8} y_{q3} \tag{67}$$

which using (56) and symmetry yields the approximate expression for Q_T:

$$Q_T = 2\left[\frac{Q_1\left(Q_2+\frac{\pi}{4}\right)^2\left(Q_3+\frac{\pi}{4}\right)\left(Q_1+\frac{\pi}{4}\right)}{2}\right]^{1/5}. \tag{68}$$

Putting numbers into this example, letting $Y_{q1}/Y_0 = y_{q1} = 1.2680$, then (66) yields $Y_{q2}/Y_0 = y_{q2} = 0.4141$ and $Y_{q3}/Y_0 = y_{q3} = 0.3075$. With these values for y_{q1}, y_{q2}, and y_{q3}, $C_3 = 225.136$, and thus $K_5 = 12\,671.6$. Using (67), $Q_1 = 0.3097$, $Q_2 = 0.9483$, and $Q_3 = 1.2772$. For this example, to find the value of Q_T, (52) is numerically solved:

$$\sin^2\left(\frac{\pi}{2}\frac{\omega}{\omega_0}\right) = 12671.6\cos^{10}\left(\frac{\pi}{2}\frac{\omega}{\omega_0}\right) \tag{69}$$

to obtain $\omega_2/\omega_0 = 1.25$ and $\omega_1/\omega_0 = 0.75$, which correspond to $Q_T = 2.0$ using (51). Using the approximation given by (56), yields $Q_T = 2.0201$. Table II summarizes the results for this example five-section filter.

IV. TABLES OF RESONATOR Q VALUES FOR A MAXIMALLY FLAT RESPONSE

This section presents tables of individual resonator Q values needed to yield a maximally flat response. The tables are organized around the number of resonant sections and Q_T. Each table corresponds to a different number of resonant sections, i.e., Table III is used for designing a two-section filter ($n = 2$), Table IV is for three resonant sections ($n = 3$), and so on from Tables V to XIII for ten resonant sections ($n = 10$). For nine and ten sections, two tables were necessary to convey all of the information. Each table is indexed by the total Q, Q_T. Since lumped-element low-pass prototype filters are typically given in terms of the g_k values [15], there are columns giving the corresponding g_k prototype values. There is also a column with the approximate value of Q_T, as calculated using (56). Finally, a column is provided for the values of 10 $\log_{10} K_n$, which was the value used by Mumford [4].

These tables are arranged around the needs of the designer. The idea being that the designer chooses the number of resonant sections and Q_T desired. The tables return the

TABLE III
QD AND g_k VALUES FOR $n = 2$

Q_T	Q_1	Q_T App.	$10\log_{10} K_n$	g_1
0.6	0.09696	0.41366	-11.14	0.3232
0.8	0.2414	0.70407	-1.899	0.6035
1	0.3757	0.93400	3.010	0.7513
2	1.056	1.9727	16.00	1.056
3	1.752	2.9824	23.18	1.168
5	3.157	4.9896	32.12	1.263
7	4.567	6.9926	37.98	1.305
10	6.686	9.9948	44.19	1.337
15	10.22	14.997	51.24	1.363
20	13.75	19.997	56.24	1.375
30	20.82	29.998	63.28	1.388
40	27.89	38.999	68.28	1.395
50	34.96	49.999	72.16	1.399
100	70.32	99.999	84.20	1.406

TABLE IV
QD AND g_k VALUES FOR $n = 3$

Q_T	Q_1	Q_2	Q_T App.	$10\log_{10} K_n$	g_1	g_2
0.6	0.04745	0.09489	0.51818	-10.84	0.1582	0.3163
0.8	0.1244	0.2488	0.77653	-0.2960	0.3110	0.6220
1	0.2048	0.4095	0.98954	6.021	0.4095	0.8190
2	0.6581	1.316	1.9989	24.34	0.6581	1.316
3	1.141	2.283	2.9997	34.92	0.7609	1.522
5	2.128	4.256	4.9999	48.23	0.8511	1.702
7	3.122	6.244	7.0000	57.00	0.8920	1.784
10	4.618	9.235	10.000	66.30	0.9235	1.847
15	7.114	14.23	15.000	76.86	0.9486	1.897
20	9.612	19.23	20.000	84.36	0.9612	1.923
30	14.61	29.22	30.000	94.92	0.9741	1.948
40	19.61	39.22	40.000	102.4	0.9805	1.961
50	24.61	49.22	50.000	108.2	0.9844	1.969
100	49.61	99.22	100.00	126.3	0.9922	1.984

TABLE V
QD AND g_k VALUES FOR $n = 4$

Q_T	Q_1	Q_2	Q_T App.	$10\log_{10} K_n$	g_1	g_2
0.6	0.02387	0.07030	0.57996	-10.54	0.07955	0.2343
0.8	0.06757	0.1940	0.81551	1.307	0.1689	0.4850
1	0.1188	0.3336	1.0185	9.031	0.2376	0.6673
2	0.4412	1.161	2.0121	32.69	0.4412	1.161
3	0.8016	2.052	3.0084	46.66	0.5344	1.368
5	1.549	3.873	5.0051	64.35	0.6195	1.549
7	2.306	5.710	7.0037	76.02	0.6589	1.631
10	3.448	8.473	10.003	88.40	0.6896	1.695
15	5.357	13.09	15.002	102.5	0.7143	1.745
20	7.268	17.70	20.001	112.5	0.7268	1.770
30	11.09	26.94	30.001	126.6	0.7395	1.796
40	14.92	36.18	40.001	136.6	0.7459	1.809
50	18.74	45.41	50.001	144.3	0.7498	1.817
100	37.88	91.61	100.00	168.4	0.7575	1.832

TABLE VI
QD AND g_k VALUES FOR $n = 5$

Q_T	Q_1	Q_2	Q_3	Q_T App.	$10\log_{10} K_n$	g_1	g_2	g_3
0.6	0.01217	0.04702	0.06972	0.62051	-10.23	0.04055	0.1568	0.2324
0.8	0.03769	0.1383	0.2012	0.83983	2.910	0.09423	0.3457	0.5029
1	0.07130	0.2496	0.3565	1.0363	12.04	0.1426	0.4991	0.7131
2	0.3097	0.9483	1.277	2.0201	41.03	0.3097	0.9483	1.277
3	0.5916	1.719	2.255	3.0136	58.40	0.3944	1.146	1.503
5	1.187	3.306	4.238	5.0082	80.46	0.4748	1.323	1.695
7	1.795	4.911	6.232	7.0059	95.04	0.5129	1.403	1.780
10	2.715	7.328	9.226	10.004	110.5	0.5430	1.466	1.845
15	4.254	11.37	14.22	15.003	128.1	0.5672	1.515	1.896
20	5.796	15.41	19.22	20.002	140.6	0.5796	1.541	1.922
30	8.884	23.49	29.22	30.001	158.2	0.5922	1.566	1.948
40	11.97	31.58	39.22	40.001	170.7	0.5986	1.579	1.961
50	15.06	39.67	49.22	50.001	180.4	0.6025	1.587	1.969
100	30.51	80.12	99.22	100.00	210.5	0.6102	1.602	1.984

TABLE VII
QD AND g_k VALUES FOR $n = 6$

Q_T	Q_1	Q_2	Q_3	Q_T App.	$10\log_{10} K_n$	g_1	g_2	g_3
0.6	0.006242	0.02986	0.05818	0.64910	-9.933	0.02081	0.09954	0.1939
0.8	0.02132	0.09475	0.1767	0.85645	4.513	0.05330	0.2369	0.4417
1	0.04361	0.1808	0.3234	1.0484	15.05	0.08721	0.3617	0.6468
2	0.2241	0.7631	1.209	2.0254	49.37	0.2241	0.7631	1.209
3	0.4514	1.426	2.153	3.0171	70.14	0.3010	0.9506	1.435
5	0.9425	2.804	4.068	5.0103	96.57	0.3770	1.122	1.627
7	1.449	4.203	5.993	7.0073	114.1	0.4139	1.201	1.712
10	2.216	6.313	8.886	10.005	132.6	0.4432	1.263	1.777
15	3.503	9.839	13.71	15.003	153.7	0.4671	1.312	1.828
20	4.794	13.37	18.54	20.003	168.7	0.4794	1.337	1.854
30	7.379	20.44	28.20	30.002	189.9	0.4919	1.363	1.880
40	9.965	27.51	37.86	40.001	204.8	0.4983	1.375	1.893
50	12.55	34.58	47.51	50.001	216.5	0.5021	1.383	1.901
100	25.49	69.93	95.81	100.00	252.6	0.5098	1.399	1.916

TABLE VIII
QD AND g_k VALUES FOR $n = 7$

Q_T	Q_1	Q_2	Q_3	Q_4	Q_T App.	$10\log_{10} K_n$	g_1	g_2	g_3	g_4
0.6	0.003214	0.01836	0.04416	0.05802	0.67033	-9.632	0.01071	0.06120	0.1472	0.1934
0.8	0.01215	0.06351	0.1427	0.1827	0.86852	6.116	0.03038	0.1588	0.3568	0.4567
1	0.02697	0.1291	0.2719	0.3397	1.0571	18.06	0.05394	0.2581	0.5438	0.6793
2	0.1655	0.6141	1.084	1.270	2.0293	57.72	0.1655	0.6141	1.084	1.270
3	0.3526	1.188	1.960	2.251	3.0196	81.88	0.2351	0.7917	1.307	1.501
5	0.7673	2.394	3.745	4.236	5.0117	112.7	0.3069	0.9575	1.498	1.695
7	1.199	3.623	5.539	6.230	7.0084	133.1	0.3426	1.035	1.583	1.780
10	1.856	5.480	8.237	9.225	10.006	154.7	0.3713	1.096	1.647	1.845
15	2.961	8.588	12.74	14.22	15.004	179.4	0.3948	1.145	1.698	1.896
20	4.070	11.70	17.24	19.22	20.003	196.8	0.4070	1.170	1.724	1.922
30	6.291	17.93	26.25	29.22	30.002	221.5	0.4194	1.195	1.750	1.948
40	8.514	24.16	35.26	39.22	40.001	239.0	0.4257	1.208	1.763	1.961
50	10.74	30.40	44.27	49.22	50.001	252.5	0.4295	1.216	1.771	1.969
100	21.86	61.57	89.31	99.22	100.00	294.7	0.4372	1.231	1.786	1.984

corresponding individual Q resonator values for the maximally flat filter. With the individual resonator Q values, the designer is free to choose the resonator that is most appropriate for the given application by relating the individual resonator Q values to the parameters of the resonator.

V. THE LUMPED-ELEMENT PROTOTYPE METHOD

The method currently used for generating maximally flat quarter-wavelength-coupled filters is briefly described, and it is pointed out why this method is only approximately correct.

The LEP method begins with the lumped-element ladder network shown in Fig. 5. The form of the response of this network is

$$\frac{P_0}{P_L} = (1 + \omega^{2n}). \tag{70}$$

The corresponding normalized element values ($g_k = L_k$ or C_k) are given by [3]

$$g_k = 2 \sin\left[\frac{(2k-1)\pi}{2n}\right], \qquad k = 1, 2, \cdots, n. \tag{71}$$

TABLE IX
QD AND g_k VALUES FOR $n = 8$

Q_T	Q_1	Q_2	Q_3	Q_4	Q_T App.	$10\log_{10} K_n$	g_1	g_2	g_3	g_4
0.6	0.001657	0.01103	0.03156	0.05111	0.68671	-9.331	0.005523	0.03677	0.1052	0.1704
0.8	0.006956	0.04191	0.1098	0.1686	0.87768	7.719	0.01739	0.1048	0.2745	0.4214
1	0.01679	0.09128	0.2196	0.3209	1.0636	21.07	0.03358	0.1826	0.4392	0.6417
2	0.1241	0.4962	0.9503	1.232	2.0321	66.06	0.1241	0.4962	0.9503	1.232
3	0.2802	0.9961	1.755	2.193	3.0214	93.62	0.1868	0.6641	1.170	1.462
5	0.6364	2.062	3.399	4.140	5.0129	128.8	0.2546	0.8247	1.359	1.656
7	1.012	3.153	5.054	6.096	7.0092	152.1	0.2890	0.9009	1.444	1.742
10	1.585	4.805	7.542	9.033	10.006	176.8	0.3171	0.9610	1.509	1.807
15	2.552	7.571	11.70	13.93	15.004	205.0	0.3402	1.010	1.559	1.858
20	3.523	10.34	15.85	18.87	20.003	225.0	0.3523	1.034	1.585	1.884
30	5.469	15.89	24.16	28.64	30.002	253.1	0.3646	1.060	1.611	1.909
40	7.418	21.45	32.48	38.45	40.002	273.1	0.3709	1.072	1.624	1.922
50	9.367	27.00	40.79	48.26	50.001	288.6	0.3747	1.080	1.632	1.930
100	19.12	54.78	82.36	97.30	100.00	336.8	0.3824	1.096	1.647	1.946

TABLE X
QD FOR $n = 9$

Q_T	Q_1	Q_2	Q_3	Q_4	Q_5	Q_T App.	$10\log_{10} K_n$
0.6	0.0008545	0.006510	0.02163	0.04154	0.05113	0.69972	-9.030
0.8	0.003990	0.02732	0.08177	0.1452	0.1736	0.88487	9.322
1	0.01050	0.06410	0.1731	0.2858	0.3327	1.0688	24.08
2	0.09396	0.4028	0.8256	1.151	1.268	2.0344	74.40
3	0.2255	0.8416	1.562	2.071	2.250	3.0229	105.4
5	0.5354	1.791	3.073	3.935	4.236	5.0137	144.9
7	0.8659	2.769	4.597	5.808	6.230	7.0098	171.1
10	1.374	4.253	6.889	8.623	9.225	10.007	198.9
15	2.232	6.740	10.71	13.32	14.22	15.005	230.6
20	3.095	9.234	14.54	18.01	19.22	20.003	253.1
30	4.827	14.23	22.20	27.41	29.22	30.002	284.8
40	6.561	19.22	29.86	36.81	39.22	40.002	307.3
50	8.296	24.22	37.52	46.20	49.22	50.001	324.7
100	16.98	49.22	75.82	93.19	99.22	100.00	378.9

TABLE XI
g_k VALUES FOR $n = 9$

Q_T	g_1	g_2	g_3	g_4	g_5
0.6	0.002850	0.02170	0.07210	0.1385	0.1704
0.8	0.009974	0.06831	0.2044	0.3631	0.4340
1	0.02010	0.1282	0.3461	0.5717	0.6655
2	0.09396	0.4028	0.8256	1.151	1.268
3	0.1503	0.5610	1.041	1.381	1.500
5	0.2142	0.7165	1.229	1.574	1.694
7	0.2474	0.7913	1.313	1.660	1.780
10	0.2748	0.8506	1.378	1.725	1.845
15	0.2976	0.8987	1.429	1.776	1.896
20	0.3095	0.9234	1.454	1.801	1.922
30	0.3218	0.9485	1.480	1.827	1.948
40	0.3280	0.9612	1.493	1.840	1.961
50	0.3318	0.9689	1.501	1.848	1.969
100	0.3395	0.9844	1.516	1.864	1.984

The LEP is transformed to a bandpass prototype by using the low-pass-to-bandpass transformation in (2), which yields the response given in (3). To use quarter-wavelength coupling, it is necessary to replace each series bandpass resonator with a shunt bandpass resonator and a section of quarter-wavelength transmission line on each side of the resonator. This equivalence was originally presented by Kuroda [16], and the resulting configuration is shown in Fig. 6. The Q of each resonant section is related to g_k in (71) by

$$Q_k = \frac{g_k}{2} Q_T \tag{72}$$

and, in terms of QD, each individual resonator has a Q given by [7]

$$Q_k = Q_T \sin\left[\frac{(2k-1)\pi}{2n}\right], \qquad k = 1, 2, \cdots, n. \tag{73}$$

Unfortunately, (73) does not result in a maximally flat response and does not give the correct total Q. The reason is that the quarter-wavelength sections of transmission lines contribute to the filter response. For high total Q filters, this effect is not as noticeable, but for low total Q filters, this selectivity causes ripples in the passband. In addition, since a quarter-wavelength section of transmission line has selectivity, it adds to the Q of the filter. Thus, the total Q found using (73) is not the correct total Q value. It approaches the correct value only at high total Q values. The QD method corrects both of these problems because it implicitly accounts for the selectivity of the quarter-wavelength coupling lines. The results using the LEP method versus the QD method are demonstrated by the example in Section VI.

TABLE XII
QD FOR $n = 10$

Q_T	Q_1	Q_2	Q_3	Q_4	Q_5	Q_T App.	$10\log_{10} K_n$
0.6	0.0004413	0.003788	0.01436	0.03190	0.04645	0.71031	-8.729
0.8	0.002291	0.01762	0.05947	0.1196	0.1643	0.89067	10.93
1	0.006578	0.04473	0.1341	0.2457	0.3206	1.0729	27.09
2	0.07172	0.3283	0.7144	1.055	1.244	2.0362	82.74
3	0.1833	0.7157	1.388	1.926	2.213	3.024	117.1
5	0.4555	1.569	2.780	3.692	4.174	5.0144	161.0
7	0.7499	2.453	4.185	5.468	6.143	7.0103	190.1
10	1.205	3.797	6.300	8.136	9.102	10.007	221.0
15	1.976	6.053	9.831	12.59	14.04	15.005	256.2
20	2.753	8.316	13.36	17.04	18.97	20.004	281.2
30	4.312	12.85	20.43	25.95	28.85	30.002	316.4
40	5.873	17.39	27.50	34.86	38.73	40.002	341.4
50	7.436	21.92	34.57	43.77	48.60	50.001	360.8
100	15.25	44.62	69.93	88.32	97.98	100.00	421.0

TABLE XIII
g_k VALUES FOR $n = 10$

Q_T	g_1	g_2	g_3	g_4	g_5
0.6	0.001471	0.01263	0.04786	0.1063	0.1548
0.8	0.005728	0.04406	0.1487	0.2990	0.4108
1	0.01316	0.08947	0.2682	0.4913	0.6412
2	0.07172	0.3283	0.7144	1.055	1.244
3	0.1222	0.4771	0.9256	1.284	1.475
5	0.1822	0.6274	1.112	1.477	1.670
7	0.2143	0.7008	1.196	1.562	1.755
10	0.2410	0.7593	1.260	1.627	1.820
15	0.2635	0.8070	1.311	1.678	1.872
20	0.2753	0.8316	1.336	1.704	1.897
30	0.2874	0.8566	1.362	1.730	1.923
40	0.2937	0.8692	1.375	1.743	1.936
50	0.2974	0.8769	1.383	1.751	1.944
100	0.3051	0.8924	1.399	1.766	1.960

VI. EXAMPLE COMPARING THE LEP AND QD METHODS

This section compares the QD and LEP methods through an example of designing a maximally flat five-section filter. In doing so, it is demonstrated how to implement both methods. Also, the results from both methods are compared and it is pointed out that the LEP method yields a response that is not truly maximally flat and a total Q which deviates significantly from its intended value.

A. Design Specifications

Following the example provided in Section III-H, a five-section quarter-wavelength-coupled filter is created which has a total Q, $Q_T = 2.0$. For simplicity, quarter-wavelength shorted-stub resonators that resonate at $f_0 = 1$ GHz are used. This filter will be implemented with source and load impedances of 50 Ω and with main transmission-line impedance $Z_0 = 50$ Ω. Table XIV lists the design parameter specifications. The circuit layout is shown in Fig. 7.

To build the filter using the LEP method, (73) is used to calculate the appropriate values for Q_i. Then, (43) is used to convert these Q_i values into stub impedance values. Table XV lists the stub impedance values calculated using the LEP method.

For the QD method, Table VI is used with the index value of $Q_T = 2.0$ to find each Q_i. Note that these Q_i values match those given in Table II from Section III-H. With these Q_i values, (43) is again used to calculate the stub impedance values. Table XVI gives the stub impedance values calculated using the QD method.

TABLE XIV
DESIGN SPECIFICATIONS

Parameter	Value
n	5
Q_T	2.0
f_0	1 GHz
$Z_S = Z_L = Z_0$	50 Ω

TABLE XV
STUB IMPEDNACE VALUES USING THE LEP METHOD

i	Q_i	Z_{qi}
1	0.6180	31.7700
2	1.6180	12.1351
3	2.0000	9.81748

TABLE XVI
STUB IMPEDANCE UING THE QD METHOD

i	Q_i	Z_{qi}
1	0.3097	63.3989 Ω
2	0.9483	20.7058 Ω
3	1.2772	15.3740 Ω

TABLE XVII
Q_i COMPARISON

	LEP Method	QD Method
Q_1	0.6180	0.3097
Q_2	1.6180	0.9483
Q_3	2.0000	1.2772
Q_T	2.7855	2.0

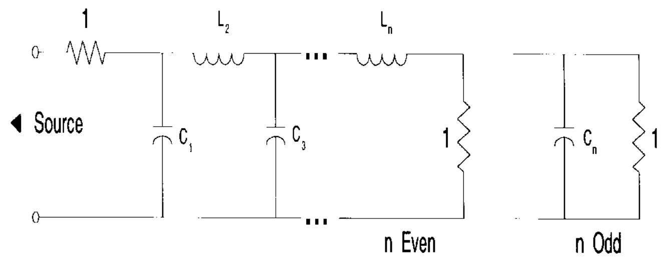

Fig. 5. General low-pass lumped-element filter.

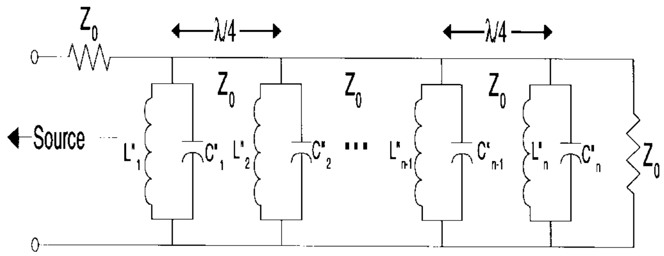

Fig. 6. General lumped-element quarter-wavelength-coupled filter.

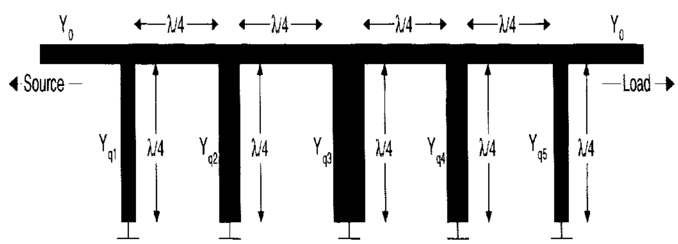

Fig. 7. Five-section filter using quarter-wavelength shorted-stubs.

B. Comparison of the Results

Table XVII compares the individual resonator Q values for each method. Also, Table XVII shows the value of Q_T found from the actual filter response. These values were found by locating the upper and lower 3-dB frequencies and then using (51) to calculate Q_T. Figs. 8 and 9 compare the filter responses of the two methods. Notice that in Fig. 8 the QD method is completely flat in the passband and that the LEP method has ripples in the passband. Fig. 9 shows the upper and lower 3-dB frequencies used to calculate Q_T in Table XVII. Overall, the response using the QD method is practically exact whereas the response using the LEP method significantly deviates from the ideal maximally flat response.

VII. Conclusion

This paper has presented a technique for accurately designing maximally flat quarter-wavelength-coupled filters. The current method used—the LEP method—which is based on the lumped-element low-pass prototype network, inaccurately predicts the value for the total Q of the filter and yields a response that is not truly maximally flat. In this paper, the technique presented (called the QD method) corrects both of these problems. With the QD method, the designer chooses the number of resonant sections and the total Q for the filter. Using the tables provided or following the method presented, the distribution of individual Q values is returned to the designer. With this, the actual choice of resonator is arbitrary. A five-section filter example was used to demonstrate both the validity of the QD method and the problems with the LEP method.

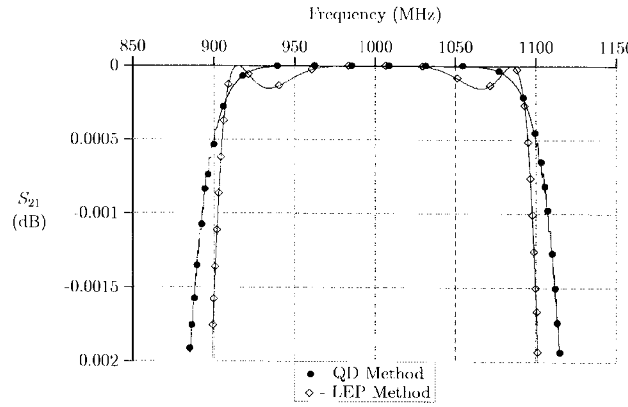

Fig. 8. Comparison of the passbands for the QD and LEP methods.

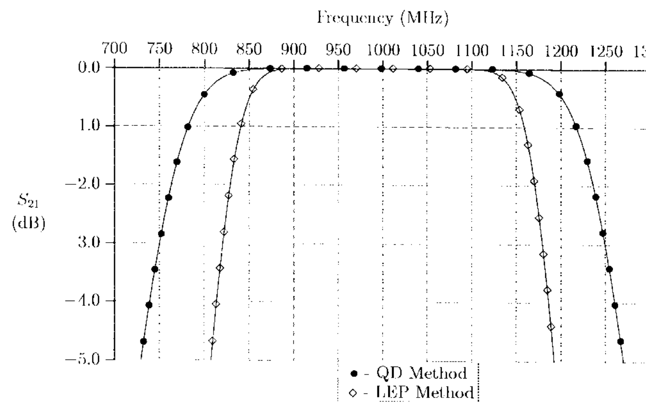

Fig. 9. Comparison of the 3-dB points for the QD and LEP methods.

References

[1] W. R. Bennett, U.S. Patent 1 849 656, Mar. 15, 1932.
[2] W. W. Mumford, "Maximally flat filters in waveguide," *Bell Syst. Tech. J.*, vol. 277, pp. 684–713, Oct. 1948.
[3] G. L. Matthaei, L. Young, and E. M. T. Jones, *Microwave Filters, Impedance Matching Networks, and Coupling Structures.* New York: McGraw-Hill, 1964.
[4] W. W. Mumford, "Tables of stub admittances for maximally flat filters using shorted quarter-wave stubs," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-13, pp. 695–696, Sept. 1965.
[5] H. J. Riblet, "General synthesis of quarter-wave impedance transformers," *IRE Trans. Microwave Theory Tech.*, vol. MTT-5, pp. 36–43, Jan. 1957.
[6] J. R. Griffin and W. T. Joines, "The general transfer matrix for networks with open-circuited or short-circuited stubs," *IEEE Trans. Circuit Theory*, vol. CT-16, pp. 373–376, Aug. 1969.
[7] P. Rizzi, *Microwave Engineering Passive Circuits.* Englewood Cliffs, NJ: Prentice-Hall, 1988.
[8] W. T. Joines and J. R. Griffin, "On using the Q of transmission lines," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-16, pp. 258–260, Apr. 1968.
[9] C. G. Montgomery, R. H. Dicke, and E. M. Purcell, *Principles of Microwave Circuits.* New York: McGraw-Hill, 1948, ch. 7, pp. 230–231.
[10] J. M. Drozd and W. T. Joines, "Determining Q using S-parameter data," *IEEE Trans. Microwave Theory Tech.*, vol. 44, pp. 2123–2127, Nov. 1996.
[11] E. L. Ginzton, *Microwave Measurements.* New York: McGraw-Hill, 1957, ch. 9, pp. 391–431.

[12] M. Sucher and J. Fox, *Handbook of Microwave Measurement*, 3rd ed. New York: Wiley, 1963, vol. II, pp. 417–491.
[13] D. Kajfez and E. J. Hwan, "Q-factor measurement with network analyzer," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-32, pp. 666–670, July 1984.
[14] E.-Y. Sun and S.-H. Chao, "Unloaded Q measurement: The critical-points method," *IEEE Trans. Microwave Theory Tech.*, vol. 43, pp. 1983–1986, Aug. 1995.
[15] A. I. Zverev, *Handbook of Filter Synthesis.* New York: Wiley, 1967.
[16] K. Kuroda, "Derivation methods of distributed constant filters from lumped constant filters," in *Joint Meeting Kansai Branch Inst. Elect. Commun., Elect., Illumination Eng. Japan*, (in Japanese), Oct. 1952, p. 32.

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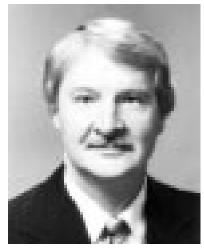

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